On the time inhomogeneous skew Brownian motion

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Abstract

This paper is devoted to the construction of a solution for the "Inhomogenous skew Brownian motion" equation, which first appeared in a seminal paper by Sophie Weinryb, and recently, studied by Étoré and Martinez. Our method is based on the use of the Balayage formula. At the end of this paper we study a limit theorem of solutions.

Keywords:
Skew Brownian motion; Local times; Stochastic differential equation, balayage formula, Skorokhod problem.

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1 Introduction

The skew Brownian motion appeared in the 70 in the seminal work [11] of Itô and McKean as a natural generalization of the Brownian motion. It is a process that behaves like a Brownian motion except that the sign of each excursion is chosen using an independent Bernoulli random variable of parameter α.

As shown in [10], this process is a strong solution to some stochastic differential equation (SDE) with singular drift coefficient:

$$X_t \quad = \quad x + B_t + (2\alpha - 1)L_t^0(X) \tag{1}$$

where $\alpha \in (0, 1)$ is the skewness parameter , $x \in \mathbb{R}$, and $L_t^0(X)$ stands for the symmetric local time at 0.

The reader may find many references concerning the homogeneous skew Brownian motion and various extensions in the literature. We cite Walsh [21], Harisson and Shepp [10], LeGall [12] and Ouknine [16].

A related stochastic differential equation, introduced by Weinryb [23] is:

$$X_t^\alpha = x + B_t + \int_0^t (2\alpha(s) - 1)\, dL_s^0(X^\alpha), \quad t \geq 0 \tag{2}$$

where $(B_t)_{t\geq 0}$ a standard Brownian motion on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\alpha : \mathbb{R}^+ \to [0, 1]$ is a Borel function and $L^0(X^\alpha)$ stands for the symmetric local time at 0 of the unknown process X^α.

The process X^α will be called "time inhomogeneous skew Brownian motion" (ISBM in short). Of course, this equation is an extenstion of the skew Brownian motion.

In [23], it was shown that there is pathwise uniqueness for equation (2) but in [23] the local time appearing in the equation is standard right sided local time, so that the function α is supposed to take values in $]-\infty, 1/2]$. As is well known, weak existence combined with pathwise uniqueness, establishes existence and uniqueness of a strong solution to (2), via the classical result of Yamada and Watanabe. So, Our purpose in this paper, is to give an explicit construction of the solution of (2) by approximating the function α by a sequence of piecewise constant functions (α_n). In order to treat the simple case of a given piecewise constant, we are inspired by a construction

given by Étoré and Martinez [9], but our proof is totally different. Instead of trying to show that our construction preserves the Markov property and that the constructed process satisfies (2), we use the Balayage formula: the key of "first order calculus". After its first appearance in Azéma and yor [1], it was later studied extensively in a series of papers as [5],[20] and [24]. Note that the point of our departure in this sense, is an interesting observation of Prokaj (see Proposition 3 [17]), his work is strongly related to the work of Gilat. [6].

In [6], the author proved that every nonnegative submartingale is equal in law to the absolute value of a martingale M. Barlow in [3] gives an explicit construction of the martingale M but for a remarkable class of submartingales. We will show that this result of Barlow is a direct consequence of the Balayage formula.

The paper is organized as follows: The second section, we start it with the progressive version of the Balayage formula and we show how to deduce from it a generalization of the observation of Prokaj, in the same section we give a simple proof of the result of Barlow [3]. Section 3 is devoted to the construction of a weak solution of equation (2) with a piecewise constant function. Extension of the above result to general case where α is a Borel function is the subject of same section. At end of this work, we study the stability of the solutions of equation (2) by using the Skorokhod representation theorem. These result was obtained by Étoré and Martinez [9] but under some monotonicity assumptions.

1.1 Preliminaries

The ISBM has many interesting and sometimes unexpected properties see Etoré and Martinez [9]. So, The main facts that we use in this paper will be summarized in this section.

Notation

For a given semimartingale X, we denote by $L_t^0(X)$ its symmetric local time at level 0. The expectation $\mathbb{E}^x$ refers to the probability measure $\mathbb{P}^x$ under which $X_0^\alpha = x$, $\mathbb{P}$-a.s.

If k is a measurable bounded process, ${}^p k$ will be denote the predictable projection of k.

(σ_t) is the shift operator acting on time dependent functions as follows:

$\alpha \circ \sigma_t(s) = \alpha(t+s)$.

Proposition 1.1 (see [19]) *Assume* (1) *has a weak solution* X^α. *Then under* $\mathbb{P}^0$,

$$(|X_t^\alpha|)_{t\geq 0} \overset{\mathcal{L}}{\sim} (|B_t|)_{t\geq 0}.$$

In the introductory article [23], it is shown that there is pathwise uniqueness for the weak solutions of equation (2).

Theorem 1.1 *Pathwise uniqueness holds for the weak solutions of equation (2).*

Definition 1.1 *For* $t > s$, $x, y \in \mathbb{R}$, *we set*

$$\begin{aligned} p^\alpha(s,t;x,y) &= \int_0^{t-s} \frac{1+\mathrm{sgn}(y)(2\alpha-1)\circ\sigma_s(u)}{2} \frac{|y|}{\pi} \frac{e^{-\frac{y^2}{2(t-(s+u))}}}{\sqrt{u}(t-s-u)^{3/2}} e^{-x^2/2u} du \\ &+ \frac{1}{\sqrt{2\pi(t-s)}} \exp -\frac{(y-x)^2}{2(t-s)} - \exp -\frac{(y+x)^2}{2(t-s)} \mathbf{1}_{\{xy>0\}}. \end{aligned}$$

Proposition 1.2 (See [9]) *Let* X^α *a strong solution of (2) corresponding to the Brownian motion* B.

i) The process X^α *is a Markov process*

ii) For all $x, y \in \mathbb{R}$ *we have*

$$\mathbb{P}^{s,x}(X_t^\alpha \in dy) = p^\alpha(s,t,x,y)dy. \tag{3}$$

The next results shows a Kolmogorov's continuity criterion for X^α uniform with respect to the parameter function $\alpha(.)$.

Proposition 1.3 (see [9]) *There exists a universal constant* $C > 0$ *such that for all* $\epsilon \geq 0$ *and* $t \geq 0$

$$\mathbb{E}^x|X_{t+\epsilon}^\alpha - X_t^\alpha|^4 \leq C\epsilon^2.$$

2 Main results

2.1 Some results in the case where α is constant

If M is martingale, k is a predictable process such that $\int_0^t k_s^2 d\langle M, M\rangle_s < \infty$, we saw how stochastic integration allows to construct a new martingale

namely $\int_0^t k_s dM_s$ with increasing process $\int_0^t k_s^2 d\langle M, M\rangle_s$. The study of the analogous for the local time at 0 is strongly related to the first order calculus(see e.g.[1]).

At the beginning, let $Y = (Y_t,\ t \geq 0)$ be a continuous $\mathcal{F}_t$-semimartingale

issued from zero. For every $t > 0$ we define

$$\gamma_t = sup\{s \leq t\ :\ Y_s = 0\},$$

with the convention $\sup(\emptyset) = 0$, hence in particular $\gamma_0 = 0$. The random variables γ_t are clearly not stopping times since they depend on the future. Before stating and proving our main theorem, we shall need a powerful result (see [8])

Proposition 2.1 *(Balayage Formula)*

(i) Let Y be a continuous semimartingale, if k is a bounded progressive process, then

$$k_{\gamma_t} Y_t = k_0 Y_0 + \int_0^t {}^p k_{\gamma_s}\, dY_s + R_t,$$

where R is a process of bounded variation, adapted, continuous such that the measure dR_s is carried by the set $\{Y_s = 0\}$.

Remark 2.1 *The last zero of continuous processes plays an essential role in the balayage formula: this fact is quite surprising since such random time is not a stopping time and hence falls outside the domain of applications of the classical theorems in stochastic analysis.*

Let W be a Wiener process, it is well known (see. e.g. [19]) that

$$|W_t| \quad = \quad \beta_t + \sup(-\beta_s) \tag{4}$$

where $\beta = \int \operatorname{sign}(W_s) dW_s$ is another Wiener process. In other words, starting with β, the skorokhod reflection of β defined as the right hand side of (4) can be unfolded to a Wiener Process W. A similar statement was proved by Prokaj for continuous semimartingales in [17]. Precisely, If U is continuous semimartingale starting from zero and Y denotes the Skorokhod reflection of U. Prokaj has showed that Y can be represented as the reflection $|Y'|$ of

an appropriate semimartingale Y' related to Y via the Tanaka equation

$$Y_t = |Y'_t| \quad \text{and} \quad Y'_t = \int_0^t \text{sign}(Y'_s) dU_s.$$

The proof of this result is based on his key observation corresponding to the case $\alpha = \frac{1}{2}$ in Proposition 2.2 below.
In what follows, we show that this observation is a direct consequence of the balayage formula. We review here this method from Prokaj:

Put $\mathfrak{z} = \{t \geq 0 \; : \; Y_t = 0\}$, this set cannot be ordered. However, the set $\mathbb{R}^+ \setminus \mathfrak{z}$ can be decomposed as a countable union $\cup_{n\in\mathbb{N}} J_n$ of intervals J_n. Each interval J_n corresponds to some excursion of Y. That is if $J_n =]g_n, d_n[$,

$$Y_t \neq 0 \text{ for } t \in]g_n, d_n[, \text{ and } Y_{g_n} = Y_{d_n} = 0.$$

At each J_n we associate a Bernoulli random variable ξ_n which is independent from any other random variables and such that $\mathbb{P}[\xi_n = 1] = \alpha$ and $\mathbb{P}[\xi_n = -1] = 1 - \alpha$. This can be achieved by a suitable enlargement of the probability field.
Now let Z be the process given by:

$$Z_t = \sum_{n=0}^{+\infty} \xi_n \mathbf{1}_{]g_n, d_n[}(t).$$

Proposition 2.2

$$Z_t Y_t = \int_0^t Z_s dY_s + (2\alpha - 1) L_t^0(ZY)$$

Proof. To show this proposition, we use the Balayage formula stated in the first part of this paper. First, let us define a process k by :

$$k_t = \sum_{n=0}^{+\infty} \xi_n \mathbf{1}_{[g_n, d_n[}(t).$$

It is obvious that k is progressive and bounded. On other hand, we remark that

$$Z_t Y_t = k_{\gamma_t} . Y_t,$$

thanks to the Balayage formula, we have:

$$Z_t Y_t = k_{\gamma_t}.Y_t = \int_0^t {}^p(k_{\gamma_s})\, dY_s + R_t. \tag{5}$$

Using the definition of k, it is clear that ${}^p(k_{\gamma_s}) =^p (k_s) = k_{s-} = Z_s$. Thus, (5) has the form:

$$Z_t Y_t = \int_0^t Z_s\, dY_s + R_t. \tag{6}$$

To identify the process R, we use a standard approximation of the process $Z_t Y_t - \int_0^t Z_s dY_s$. For $\epsilon > 0$, let us define the following sequence of stopping times:

$$\begin{aligned} \tau_0^\epsilon &= 0 \\ \tau_{2k+1}^\epsilon &= \inf\{t > \tau_{2k}^\epsilon \ : \ |Y_t| > \epsilon\} \quad k = 0, 1, 2, ... \\ \tau_{2k+2}^\epsilon &= \inf\{t > \tau_{2k+1}^\epsilon \ : \ |Y_t| = 0\} \quad k = 0, 1, 2, ... \end{aligned}$$

Put,

$$Z_t^\epsilon = \sum_{k=0}^{+\infty} Z_t \mathbf{1}_{(\tau_{2k+1}^\epsilon, \tau_{2k+2}^\epsilon]}(t).$$

Z^ϵ is constant on every random interval of the form $(\tau_{2k+1}^\epsilon, \tau_{2k+2}^\epsilon]$. The continuity of Y ensures that Z^ϵ is of bounded variation on every compact interval. Hence, Y is Riemann-Stieltjes integrable with respect to Z^ϵ almost surely and

$$Z_t^\epsilon Y_t - Z_0^\epsilon Y_0 - \int_0^t Z_s^\epsilon\, dY_s = \int_0^t Y_s\, dZ_s^\epsilon. \tag{7}$$

As $\epsilon \to 0$ we have that $Z_t^\epsilon \to Z_t$ for all t almost surely. Since $|Z^\epsilon| \leq 1$ the convergence of Z^ϵ implies as well that

$$\int_0^t Z_s^\epsilon\, dY_s \to \int_0^t Z_s\, dY_s$$

in probability for all t. The definition of Z^ϵ entails that,

$$\int_0^t Y_s \, dZ_s^\epsilon = \sum_{\substack{k \\ \tau_{2k+1}<t}} Y_{\tau_{2k+1}^\epsilon} Z_{\tau_{2k+1}^\epsilon} = \epsilon \sum_{\substack{k \\ \tau_{2k+1}<t}} Z_{\tau_{2k+1}^\epsilon}$$

Let $N(t,\epsilon)$ be the number of upcrossing of the interval $[0,\epsilon]$. So,

$$\int_0^t Y_s \, dZ_s^\epsilon = \epsilon \sum_{l=1}^{N(t,\epsilon)} \xi_{k_l}$$

where $\{\xi_{k_l}, \ \ l = 0, 1, ..., N(t,\epsilon)\}$ is an enumeration of the $Z_{\tau_{2k+1}}^\epsilon$ values. By a Lévy's résult [14], we have $\lim_{\epsilon \to 0} \epsilon N(t,\epsilon) = \frac{1}{2} L_t^0(|Y|)$. So, we can apply the Bernoulli law of large numbers with to get:

$$\epsilon N(t,\epsilon) \frac{\sum_{l=1}^{N(t,\epsilon)} \xi_{k_l}}{N(t,\epsilon)} \to \frac{1}{2} L_t^0(|Y|).\mathbb{E}[\xi_1]$$

Since $\mathbb{E}[\xi_1] = 2\alpha - 1$, we obtain:

$$\int_0^t Y_s \, dZ_s^\epsilon \to (2\alpha - 1) \frac{1}{2} L_t^0(|Y|),$$

since Z has values in the set $\{-1, 1\}$, It is clear that

$$\frac{1}{2} L_t^0(|Y|) = \frac{1}{2} L_t^0(|ZY|) = L_t^0(ZY).$$

Hence the result. ■

2.2 Construction of continuous martingale with given absolute value

We devote this subsection to an application of Proposition 2.2. For this, we will first need to introduce a special class of submartingales whose introduction goes back to Yor [24].

Definition 2.1 *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space. A nonnegative (local) submartingale $(X_t)_{t \geq 0}$ is of class Σ, if it can be decomposed as*

$X_t = N_t + A_t$ *where* $(N_t)_{t\geq 0}$ *and* $(A_t)_{t\geq 0}$ *are* $(\mathcal{F}_t)-$*adapted processes satisfying the following assumptions:*

- $(N_t)_{t\geq 0}$ *is a continuous (local) martingale.*
- $(A_t)_{t\geq 0}$ *is continuous increasing process, with* $A_0 = 0$.
- *The measure* (dA_t) *is carried by the set* $\{t \geq 0, X_t = 0\}$

The class (Σ) contains many well-known examples of stochastic processes such as a nonnegative local martingales, $|M_t|$, M_t^+, M_t^- if M is a continuous local martingale. Other remarkable families of examples consist of a large class of recurrent diffusions on natural scale (such as some powers of Bessel processes of dimension $\delta \in (0, 2)$, see [15]) or of a function of a symmetric Lévy processes; in these cases, A_t is the local time of the diffusion process or of the Lévy process.

In [6], Gilat proved that every nonnegative submartingale Y is equal in law to the absolute value of a Martingale M. His construction, however, did not shed any light on the nature of this martingale. In the case when X is of class Σ, one could deduce more interesting result from Proposition 2.2. The proof is very simple in this case and we give it.

Proposition 2.3 *If* $X \in \Sigma$, *then there exists a continuous martingale* M *such that* $X = |M|$

Proof. In this proof, we use the same notations as above, let $\alpha = \frac{1}{2}$, by proposition 2.2, we have:

$$Z_t X_t = \int_0^t Z_s dX_s = \int_0^t Z_s dN_s + \int_0^t Z_s dA_s$$

By this argument of the support *supp* $dA_s \subset \{X_s = 0\}$, it is clear that $\int_0^t Z_s dA_s = 0$, putting $M_t = Z_t X_t$, since Z has values in $\{-1, 1\}$, we get $X_t = |M_t|$ ■

2.3 The case where α is a piecwise constant function

Thinking of the case of he proposition 2.2 in which the parameter α is constant, we will content ourselves to find the analogous of this result for a

piecewise constant function α.

Let $\{\pi : 0 = t_0 < t_1 ... < t_i ... < t_m = 1\}$ be a partition of the interval $[0, 1]$.
Let $\alpha : [0, 1] \to [-1, 1]$ be a r.c.l.l function with constant value in each interval $[t_i, t_{i+1})$. So, α has the form $\alpha(t) = \sum_{i=0}^{m} \alpha_i \mathbf{1}_{\{[t_i, t_{i+1})\}}$ where $\alpha_i \in [0, 1]$ for all $i = 0, ..., m$.

Let $(\xi_n^i)_{n\geq 0}$, $i = 0, 1, 2, ...m$ be m independent sequences of independent variables such that, $\mathbb{P}(\xi_n^i = 1) = \alpha(t_i)$ and $\mathbb{P}(\xi_n^i = -1) = 1 - \alpha(t_i)$.

Now, put

$$Z_t = \sum_{n=0}^{+\infty} \sum_{i=0}^{m} \xi_n^i \mathbf{1}_{]g_n, d_n[\bigcap[t_i, t_{i+1})}(t).$$

Proposition 2.4

$$Z_t Y_t = \int_0^t Z_s dY_s + \int_0^t (2\alpha(s) - 1) dL_s^0(ZY)$$

Proof. The proof follows the same line as the proof of proposition 2.2. For a sake of completeness, we give again a proof for the general case.
To use the Balayage formula, we must make a suitable choice of k. Let k be the process defined by

$$k_t = \sum_{n=0}^{+\infty} \sum_{i=0}^{m} \xi_n \mathbf{1}_{[g_n, d_n[\bigcap[t_i, t_{i+1})}(t).$$

This definition is in fact quite intuitive. It is obvious that k is progressive and bounded. On other hand, we remark that

$$Z_t Y_t = k_{\gamma_t}.Y_t,$$

,as in Proof of the first Proposition, it suffices to apply Balayage formula to get

$$Z_t Y_t = \int_0^t Z_s \, dY_s + A_t. \tag{8}$$

to identify the process A we use a standard approximation of the process $Z_tY_t - \int_0^t Z_s dY_s$ by using the same sequence of stopping times as in the proof of proposition 2.2. Put,

$$Z_t^\epsilon = \sum_{i=0}^{m} \sum_{\substack{k=0 \\ t_i < \tau_{2k+1} < t_{i+1}}}^{+\infty} \xi_k^i \mathbf{1}_{(\tau_{2k+1}^\epsilon, \tau_{2k+2}^\epsilon]}(t).$$

Z^ϵ is constant on every random interval of the form $(\tau_{2k+1}^\epsilon, \tau_{2k+2}^\epsilon]$. The continuity of Y ensures that Z^ϵ is of bounded variation on every compact interval. Hence, Y is Riemann-Stieltjes integrable with respect to Z^ϵ almost surely and

$$Z_t^\epsilon Y_t - Z_0^\epsilon Y_0 - \int_0^t Z_s^\epsilon \, dY_s = \int_0^t Y_s \, dZ_s^\epsilon. \tag{9}$$

As $\epsilon \to 0$ we have that $Z_t^\epsilon \to Z_t$ for all t almost surely. Thus

$$\int_0^t Z_s^\epsilon \, dY_s \to \int_0^t Z_s \, dY_s$$

By the definition of Z^ϵ,

$$\int_0^t Y_s \, dZ_s^\epsilon = \sum_{i=0}^{m} \sum_{\substack{k=0 \\ t_i < \tau_{2k+1} < t_{i+1}}}^{+\infty} Y_{\tau_{2k+1}^\epsilon} Z_{\tau_{2k+1}^\epsilon} = \epsilon \sum_{i=0}^{m} \sum_{\substack{k=0 \\ t_i < \tau_{2k+1} < t_{i+1}}}^{+\infty} Z_{\tau_{2k+1}^\epsilon}$$

Let $N(t_i, t_{i+1}, \epsilon)$ be the number of upcrossing of the interval $[0, \epsilon]$ between t_i and t_{i+1}. So,

$$\int_0^t Y_s \, dZ_s^\epsilon = \epsilon \sum_{i=0}^{m} \sum_{l=1}^{N(t_i, t_{i+1}, \epsilon)} \xi_{k_l}^i$$

where $\{\xi_{k_l}^i, \ \ l = 0, 1, ..., N(t_i, t_{i+1}, \epsilon)\}$ is an enumeration of the $Z_{\tau_{2k+1}}^\epsilon$ values between t_i and t_{i+1}. Thus,

$$\int_0^t Y_s \, dZ_s^\epsilon \quad = \quad \epsilon \sum_{i=0}^{m} \left[\sum_{l=1}^{N(0, t_{i+1}, \epsilon)} \xi_{k_l}^i - \sum_{l=1}^{N(0, t_{i+1}, \epsilon)} \xi_{k_l}^i \right]$$

We can apply the Bernoulli law of large numbers to get:

$$\epsilon N(0,t_{i+1},\epsilon)\frac{\sum_{l=1}^{N(0,t_{i+1},\epsilon)}\xi_{k_l}^i}{N(0,t_i,\epsilon)} \to \frac{1}{2}L_{t_{i+1}}^0(|Y|).\mathbb{E}[\xi_1^i] \qquad \forall i=0,1,...,m-1$$

and by the same arguments:

$$\epsilon N(0,t_i,\epsilon)\frac{\sum_{l=1}^{N(0,t_i\epsilon)}\xi_{k_l}^i}{N(0,t_i,\epsilon)} \to \frac{1}{2}L_{t_i}^0(|Y|).\mathbb{E}[\xi_1^i] \qquad \forall i=1,2,...,m$$

Since $\mathbb{E}[\xi_1^i]=2\alpha(t_i)-1$, we get:

$$\int_0^t Y_s\,dZ_s^\epsilon \to \frac{1}{2}\sum_{i=1}^m(2\alpha(t_i)-1)(L_{t_{i+1}}(|Y|)-L_{t_i}(|Y|)) = \frac{1}{2}\int_0^t(2\alpha(s)-1)dL_s^0(|Y|),$$

cleary this implies

$$\int_0^t Y_s\,dZ_s^\epsilon \to \int_0^t(2\alpha(s)-1)dL_s^0(ZY).$$

Hence the result. ■

3 Construction of the inhomogenous SBM

3.1 Construction of the inhomogenous SBM with the piece-wise constant coefficient α

In this section, we give a construction of a weak solution of (2) obtained by an application of the Proposition 2.4. We use the same notations and assumptions of the subsection 2.3.
The construction is the following: Let B a standard $(\mathcal{F}_t)-$Brownian motion. We define a process X^α by putting

$$\forall t\geq 0 \quad X_t^\alpha(\omega)=Z_t(\omega).|B_t(\omega)|, \tag{10}$$

where the process Z is defined by $Z_t=\sum_{n=0}^{+\infty}\sum_{i=0}^m\xi_n^i\mathbf{1}_{]g_n,d_n[\bigcap[t_i,t_{i+1})}(t)$, where $\{]g_n,d_n[\}$ is a countable unions of disjoint intervals which covers the set

$\{s \geq 0,\ B_s \neq 0\}$. Consequently, we have the following theorem

Theorem 3.1 *The process X^α is a weak solution of (2) with parameter α and starting from 0.*

Proof. By Proposition 2.4,

$$X_t^\alpha = \int_0^t Z_s\, d|B_s| + \int_0^t (2\alpha(s)-1) dL_s^0(X^\alpha).$$

Tanaka's formula implies that:

$$\begin{aligned} X_t^\alpha &= \int_0^t Z_s \mathrm{sign}(B_s) dB_s + \int_0^t Z_s\, dL_s^0(B) + \int_0^t (2\alpha(s)-1) dL_s^0(X^\alpha) \\ &= \int_0^t Z_s d\beta_s + \int_0^t (2\alpha(s)-1) dL_s^0(X^\alpha) \end{aligned}$$

where $\beta = \int sg(B_s) dB_s$ is a Browninan motion. In the last line we have used the fact that the measure $dL_s^0(B)$ is carried by the set $S = \{s\ , B_s = 0\}$ and that the process Z is defined on the complementary of the set S, hence $\int_0^t Z_s\, dL_s^0(B) = 0$. Obviously, the process $(\int_0^t Z_s\, d\beta_s)$ is a continuous local martingale. Since $\int_0^t Z_s^2 ds = t$, we deduce that $(\int_0^t Z_s\, dB_s)$ is in fact a Brownian motion, ensuring that X^α satisfies (2). ■

3.2 Construction of ISBM with a borel function α

Now, as a natural extension of the construction in the last subsection, we have the following theorem

Theorem 3.2 *Let $\alpha : \mathbb{R}^+ \to [0,1]$ a Borel function and B a standard Brownian motion. For any fixed $x \in \mathbb{R}$, there exists a unique (strong) solution to (2). It is a strong Markov process with transition function $p^\alpha(s,t,x,y)$.*

The main tool used in the proof is the Skorokhod representation theorem given by the following:

Lemma 3.1 *Let (S, ρ) be a complete separable metric space, $\{\mathbb{P}_n\ , n \geq 1\}$ and $\mathbb{P}$ be probability measures on $(S, \mathcal{B}(S))$ such that $\mathbb{P}_n \xrightarrow{n \to +\infty} \mathbb{P}$. Then on a probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$, we construct $S-$valued random variables X_n, $n = 1, 2, ...$ and X, such that:*

(i) $\mathbb{P}_n = \widehat{\mathbb{P}}_{X_n}$ $n = 1, 2, ...$ *and* $\mathbb{P} = \widehat{\mathbb{P}}_X$

(ii) X_n *converge to* X, $\widehat{\mathbb{P}}$ *almost surely.*

We will make use of the following result, which gives a criterion for tightness of sequences of laws associated to continuous processes.

Lemma 3.2 *Let* $X^n(t)$, $n = 1, 2...$, *be a sequence of d-dimensional continuous processes satisfying two conditions:*

(i) There exist positive positive constants, M *and* l *such that*

$$\mathbb{E}(|X_0^n|^l) \leq M \quad \textit{for every } n = 1, 2, ..$$

(ii) There exist positive positive constants, p, q, M_k, $k = 1, 2, ...$ *such that:*

$$\mathbb{E}(|X_t^n - X_s^n|^p) \leq M_k |t - s|^{1+q} \textit{ for every } n \textit{ and } t, s \in [0, k].$$

Then there exits a subsequence $(n_k)_{k \geq 1}$, *a probability space* $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$ *and d-dimensional continuous processes* $\widehat{X}_{n_k}$, $k = 1, 2, ...$ *and* $\widehat{X}$ *defined on it such that*

1. *The laws of* $\widehat{X}^{n_k}$ *and* X^{n_k} *coincide.*
2. $\widehat{X}_t^{n_k}$ *converges to* $\widehat{X}_t$ *uniformly on every finite time interval* $\widehat{\mathbb{P}}$ *almost surely.*

Proof of theorem 3.2. $\alpha(.)$ is a borel function, so , there exists a sequence of piecewise constant functions α_n such that $\lim_{n \to +\infty} \alpha_n(t) = \alpha(t)$, $\forall t \in [0, 1]$. Corresponding to such sequences α_n, we introduce the corresponding sequences $(X^n)_{n \geq 0}$ which are solutions to equation (2) constructed as in the beginning of this section, thus

$$X_t^n = x + B_t + \int_0^t (2\alpha_n(s) - 1) dL_s^0(X^n) \quad \forall n \in \mathbb{N}$$

By Lemma 3.2 and Proposition 1.3, It is clear that the family (X^n, B) is tight. Then there exist a probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$ and a sequence $(\widehat{X}^n, \widehat{B}^n)$ of stochastic processes defined on it such that:

- $[P.1]$ The laws of (X^n, B) and $(\widehat{X}^n, \widehat{B}^n)$ coincide for every $n \in \mathbb{N}$.

- $[P.2]$ There exists a subsequence $(n_k)_{k\geq 0}$ such that: $(\widehat{X}^{n_k}, \widehat{B}^{n_k})$ converge to $(\widehat{X}, \widehat{B})$ uniformly on every compact subset of $\mathbb{R}^+$ $\widehat{\mathbb{P}}-$a.s.

If we denote $\widehat{\mathcal{F}}_t^n = \sigma\{\widehat{X}_s^n, \widehat{B}_s^n \ ; \ s \leq t\}$ and $\widehat{\mathcal{F}}_t = \sigma\{\widehat{X}_s, \widehat{B}_s \ ; \ s \leq t\}$, then $(\widehat{B}^n, \widehat{\mathcal{F}}^n)$ and $(\widehat{B}, \widehat{\mathcal{F}})$ are Brownian motions. According to property $[P.1]$ and the fact that X_t^n satisfies equation (2), it's can be proved that

$$\mathbb{E}\left|\widehat{X}_t^n - x - \widehat{B}_t^n - \int_0^t \alpha_n(s) dL_s^0(\widehat{X}^n)\right|^2 = 0.$$

In other words, $\widehat{X}_t^n$ satisfies the SDE:

$$\widehat{X}_t^n = x + \widehat{B}_t^n + \int_0^t (2\alpha_n(s) - 1) dL_s^0(\widehat{X}^n).$$

On one hand, from the fact that $(|\widehat{X}_t^{n_k}|)_{t\geq 0} \overset{law}{=} (|\widehat{B}_t^{n_k}|)_{t\geq 0}$ and condition $[P.2]$, we deduce that $(|\widehat{X}_t|)_{t\in[0,1]} \overset{law}{=} (|\widehat{B}_t|)_{t\in[0,1]}$. Thus, $(|\widehat{X}_t|)_{t\in[0,1]}$ is a semimartingale and admits a symmetric local time process $L_\cdot(|\widehat{X}|)$.
A consequence of the Tanaka formula is that:

$$|\widehat{X}_t^{n_k}| = |x| + \int_0^t \operatorname{sign}(\widehat{X}_s^{n_k}) d\widehat{B}_s^{n_k} + L_t^0(|\widehat{X}^{n_k}|) \quad \text{(with } \operatorname{sign}(0) = 0).$$

As $|\widehat{X}|$ is a reflected Brownian motion we have

$$|\widehat{X}_t| = |x| + \widetilde{B}_t + L_t^0(|\widehat{X}|) \tag{11}$$

for some Brownian motion $\widetilde{B}$. By using property $[P.2]$ it holds that

$$\int_0^\cdot \operatorname{sign}(\widehat{X}^{n_k}) d\widehat{B}^{n_k} \xrightarrow[ucp]{L^2} \int_0^\cdot \operatorname{sign}(\widehat{X}) d\widehat{B}.$$

Thus, from the a.s uniform convergence of $(\widehat{X}_t^{n_k})_{t\in[0,1]}$ towards $(\widehat{X}_t)_{t\in[0,1]}$ and the dominated convergence for stochastic integrals (see e.g. [19]) we can see that there is a finite variation process A such that:

$$\sup_{s\in[0,1]} |L_s^0(\widehat{X}^{n_k}) - A_s| \xrightarrow[n\to\infty]{\mathbb{P}} 0,$$

and that

$$\sup_{s\in[0,1]} ||\widehat{X}^{n_k}_s| - (|x| + \int_0^s \operatorname{sign}(\widehat{X}_u)d\widehat{B}_u + A_s)| \xrightarrow[n\to\infty]{\mathbb{P}} 0.$$

Consequently,

$$|\widehat{X}_t| = |x| + \int_0^t \operatorname{sign}(\widehat{X}_s)d\widehat{B}_s + A_t \tag{12}$$

Using (11) and (12) and uniqueness of the Doob decomposition of a semi-martingale :

$$|\widehat{X}_t| = |x| + \int_0^t \operatorname{sign}(\widehat{X}_s)d\widehat{B}_s + L^0_t(|\widehat{X}|).$$

Note that we have proven that

$$\left[\sup_{s\in[0,1]} |L^0_t(\widehat{X}^{n_k}) - L^0_t(|\widehat{X}|)| > \epsilon\right] \xrightarrow[n\to\infty]{P} 0$$

Using the fact that $\widehat{X}^{n_k}$ is a Markov process with transition family (t.f.) $p^\alpha(s,t,x,y)dy$ (see 1.2) combined with property $[P.2]$ yields $\widehat{X}^{n_k} \xrightarrow{ucp} \widehat{X}$, it is obvious that $\widehat{X}$ is also a Markov process with the same t.f. Now we proceed to the proof of that $\widehat{X}$ is a solution to (2). We follow Étoré and Martinez [9] rather closely. Hence we may proceed just as in the proof of [Theorem 5.6 [9]]. In fact, by some calculus, we get

$$\mathbb{E}(\widehat{X}_t|\mathcal{F}_s) = \widehat{X}_s + \int_0^{t-s} (2\alpha - 1)\circ\sigma_s(u)\frac{e^{-\frac{|\widehat{X}_s|^2}{2u}}}{\sqrt{2\pi u}}du.$$

Note that $\widehat{X}$ is a Markov process and $|\widehat{X}|$ is a reflected Brownian motion. So that for $s < t$:

$$\mathbb{E}^0\left(\int_0^t (2\alpha(u)-1)dL^0_u(\widehat{X})|\mathcal{F}_s\right) = \int_0^s (2\alpha(u)-1)dL^0_u(\widehat{X}) + \mathbb{E}^0\left(\int_s^t (2\alpha-1)(u)dL^0_u(\widehat{X})|\mathcal{F}_s\right).$$

but,

$$\begin{aligned}
\mathbb{E}^0\left(\int_s^t (2\alpha-1)(u)dL_u^0(\widehat{X})|\mathcal{F}_s\right) &= \mathbb{E}^{\widehat{X}_s}\left(\int_s^t (2\alpha-1)\circ\sigma_s(u)dL_u^0(\widehat{X}^{(2\alpha-1)\circ\sigma_s})\right) \\
&= \mathbb{E}^{\widehat{X}_s}\left(\int_s^t (2\alpha-1)\circ\sigma_s(u)dL_u^0(|\widehat{B}|)\right) \\
&= \int_0^{t-s} (2\alpha-1)\circ\sigma_s(u)\frac{e^{-\frac{|\widehat{X}_s|^2}{2u}}}{\sqrt{2\pi u}}du.
\end{aligned}$$

Combining these facts ensures that $\{\widehat{X}_t - \int_0^t (2\alpha-1)(u)dL_u^0(\widehat{X}) \ : \ t \geq 0\}$ is a $(\mathcal{F}_t)$ local martingale. Since $\langle \widehat{X}\rangle_t = \langle |\widehat{X}|\rangle_t = t$, we deduce that, $\{\widehat{X}_t - \int_0^t (2\alpha-1)_u dL_u^0(\widehat{X}) \ : \ t \geq 0\}$ is a $(\mathcal{F}_t)$-martingale Brownian motion, ensuring that $\widehat{X}$ satisfies (2). ■

4 Stability of the solution

Another key property of the solutions of (2) is the following stability result which follows from an application of Skorokhod Representation theorem.

Theorem 4.1 *Let $\{\alpha_n(t), \alpha(t) : [0,1] \to [0,1]\}$ be a family of borel functions. Assume that $\lim_{n\to+\infty} \alpha_n(t) = \alpha(t), \quad \forall t \in [0,1]$. If we denote X^n the solution of (2) corresponding to α_n, then the following result holds: $X^n \xrightarrow[ucp]{L^2} X$, where X is the unique solution of (2) corresponding to α.*

Proof. Suppose that the conclusion of our theorem is false, then there exist a positive number δ and a subsequence n_k such that

$$\inf_{n_k} \mathbb{E}[\sup_{0\leq s\leq 1} |X_s^{n_k} - X_s|^2] \geq \delta.$$

According to lemma 3.2, the family $Z^n = (X^n, X, B)$ satisfies conditions (i) and (ii) with $p = 4$ and $q = 1$. By Skorokhod selection theorem, there exists a filtered probability space $(\Omega, \mathcal{F}', \mathbb{P}', (\mathcal{F}'_t))$ carrying a sequence of stochastic processes $\overline{Z}^{n_k}$ denoted by $\overline{Z}^n = (X'^n, \overline{X}^n, \overline{B}^n)$ with the following properties:

- [P.1.]

$$Z^n \stackrel{law}{=} \overline{Z}^n$$

- [P.2.]There exists a subsequence $(\overline{Z}^{n_k})_k$ which converges uniformly to $(X', \overline{X}, \overline{B})$.

Proceeding as in the proof of Proposition 2.4, we can see that the limiting processes satisfy the following equations.

$$X'_t = x + \overline{B}_t + \int_0^t (2\alpha(s) - 1) dL_s^0(X');$$

$$\overline{X}_t = x + \overline{B}_t + \int_0^t (2\alpha(s) - 1)(s) dL_s^0(\overline{X}).$$

In other words, X' and $\overline{X}$ solves equation (2). Thus by pathwise uniqueness, $X'$and $\overline{X}$ are indistinguishable.
By uniform integrability, it holds

$$\begin{aligned} \delta &\leq \liminf_{k \in \mathbb{N}} \mathbb{E}[\sup_{0 \leq s \leq 1} |X_t^{n_k} - X_t|^2] = \liminf_{k \in \mathbb{N}} \widehat{\mathbb{E}}[\sup_{0 \leq s \leq 1} |X_t'^{n_k} - \overline{X}_t^{n_k}|^2] \\ &\leq \widehat{\mathbb{E}}[\sup_{0 \leq s \leq 1} |X'_t - \overline{X}_t|^2] \end{aligned}$$

which is a contradiction ■

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